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The largest k-ball in a d-dimensional box

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Abstract

This paper considers d-dimensional boxes of the form $\{x \mid -a_i \leqslant x_i \leqslant a_i\}$ in E^d. It establishes a formula for the radius of the largest k-dimensional ball contained in such a d-dimensional box.

1. Introduction

This paper gives a formula for the radius $r(k,d)$ of the largest k-ball in a given d-dimensional box of the form $\{x \mid -a_i \leqslant x_i \leqslant a_i, 1 \leqslant i \leqslant d\}$. This greatly extends the known result of this type [8,9], namely, a formula for the radius of the largest disk in a 3-dimensional cube.

As a simple introduction let us consider further the problem of determining the largest disk contained in the unit cube. Any disk inside the cube determines a plane that intersects the cube in a convex polygon containing the disk. Considering disks lying in face planes provides an obvious lower bound of $r \geqslant 1/2$. The plane passing through the cube's center and containing two opposite edges determines a $1 \times \sqrt{2}$ rectangle whose largest inscribed disk again has radius $1/2$. Now consider half filling the cube with water and orienting the cube so that a long diagonal becomes vertical. The surface of the water forms a regular hexagon of side $\sqrt{2}/2$ which passes through the center of the cube and the midpoints of six edges. This hexagon contains an inscribed disk of radius $\sqrt{6}/4 \approx 0.61$. See Fig. 1, and see [1] for a photograph.

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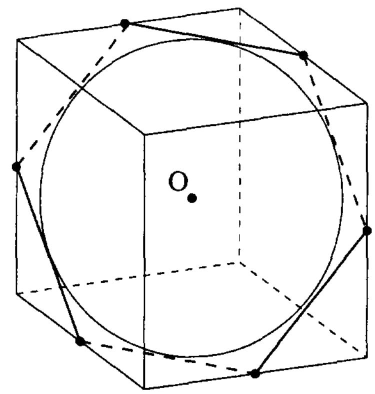

Fig. 1. A cross-section that forms a regular hexagon.

Indeed this hexagon contains a maximum radius disk. Mladenovic [8] gives a simple, direct proof of this fact, referring to an older proof in [9]. A proof follows also from our Theorem 1 by choosing $d = 3$, $k = 2$ and $a_1 = a_2 = a_3 = 1/2$.

We now give some notation and terminology. Throughout this paper, we consider a d-dimensional box $\mathcal{B}$ of dimensions $2a_1 \times 2a_2 \times \cdots \times 2a_d$. Box $\mathcal{B}$ consists of the points $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ in d-dimensional Euclidean space E^d such that $-a_i \leqslant x_i \leqslant a_i$ for each i, $1 \leqslant i \leqslant d$. Without loss of generality we may assume that the coordinate axes are labeled so that $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_d$. The box is bounded by $2d$ facets which determine hyperplanes of the form $\{\boldsymbol{x} \mid x_i = a_i\}$ or $\{\boldsymbol{x} \mid x_i = -a_i\}$. The box is centered at the origin $O = (0, 0, \ldots, 0)$.

A *k-flat* M is the translation of a k-dimensional subspace of E^d by some fixed vector. A *k-ball* of radius r and center $\boldsymbol{c}$ is the set of all points $\boldsymbol{x}$ in some k-flat M containing $\boldsymbol{c}$ such that the distance $||\boldsymbol{c} - \boldsymbol{x}||$ between $\boldsymbol{c}$ and $\boldsymbol{x}$ is at most r.

Our main result is the following.

Theorem 1. *The radius $r(k, d)$ of the largest k-ball in a d-dimensional box $\mathcal{B}$ of dimensions $2a_1 \times 2a_2 \times \cdots \times 2a_d$ is given by*

$$r(k, d) = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{d-s}^2}{k - s}},$$

where s is the smallest of the integers $0, 1, \ldots, k-1$ that satisfies

$$a_{d-s} \leqslant \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{d-s-1}^2}{k - s - 1}}.$$

(In case $k = d$ and $s = d - 1$, the empty sum $a_1^2 + a_2^2 + \cdots + a_{d-s-1}^2$ is regarded as 0.*)*

In particular, this implies the assertion of our introductory example, since in the special case $k = 2$ and $d = 3$ we get

$$r(2,3)=\begin{cases}\sqrt{\dfrac{a_1^2+a_2^2+a_3^2}{2}}, & \text{if } a_3^2\leqslant a_1^2+a_2^2,\\ \sqrt{a_1^2+a_2^2}\,, & \text{if } a_3^2> a_1^2+a_2^2.\end{cases}$$

Our k-dimensional ball in a d-dimensional box may be viewed in the context of extremal containment problems in which one seeks the largest k-dimensional object contained in a given d-dimensional object. Here, the notion of size may vary, as well as the type of both the contained object and the containing object. See [6] for some related results.

The remainder of this paper is organized as follows. Section 2 gives two preliminary lemmas. Theorem 1 is shown in Sections 3 and 4: Section 3 gives an upper bound for $r(k,d)$, which is proven to be tight [5] in Section 4. Section 5 concludes with some remarks and open problems.

2. Preliminaries

We refer the reader to any standard text on linear algebra for definitions not given in this paper. See, for example, [2].

We now state two lemmas which will be useful in the remainder of the paper. Firstly, we note that there is always a k-ball of maximum radius centered at the origin. Hence, although maximum k-balls may also be centered away from the origin, it suffices to consider only k-balls centered at the origin in order to determine the radius of the largest k-ball in box $\mathcal{B}$. The next lemma follows from a more general result presented in [6].

Lemma 1. *If a k-ball B with center $\boldsymbol{c}$ lies inside a d-dimensional box $\mathcal{B}$ symmetric with respect to the origin, then the translate $B-\boldsymbol{c}$ of this k-ball, which places its center at the origin, also lies inside $\mathcal{B}$.*

In what follows M denotes a k-dimensional flat through the origin (usually the one determined by the k-ball B), and $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k$ denotes an orthonormal basis for M, where $\boldsymbol{v}_j=(v_{j,1},v_{j,2},\ldots,v_{j,k})$. Also, M_i denotes the flat obtained by intersecting M with the hyperplane $\{\boldsymbol{x}\mid x_i=a_i\}$. Finally, we define $T_i=\sum_{j=1}^k v_{j,i}^2$.

We now give a formula for the distance from the origin to the intersection of a facet of $\mathcal{B}$ with a k-dimensional subspace (if the intersection is empty, the distance is infinite). This formula, while straightforward, does not appear explicitly in most standard vector geometry references, so we include it here for reference. A proof of a formula equivalent to this but expressed in different notation can be found in [3].

Lemma 2. *The distance from the origin to the flat M_i is given by*

$$\frac{a_i}{\sqrt{\sum_{j=1}^k v_{j,i}^2}}.$$

[5] Preliminary versions of our results appeared in [5,10]. The paper [10] claims that the upper bound for the radius of the largest k-ball is always achievable; however, the proof of Lemma 3 in [10] is not correct.

Proof. Consider the point $\boldsymbol{p} = \sum_{j=i}^{k}(a_i v_{j,i}/T_i)\boldsymbol{v}_j$. Clearly, $\boldsymbol{p} \in M$, and since its ith coordinate is a_i, $\boldsymbol{p} \in M_i$. For any other point $\boldsymbol{q} \in M_i$, computation of the inner product of $\boldsymbol{q} - \boldsymbol{p}$ with $\boldsymbol{p}$ shows that $\boldsymbol{q} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{p}$. Hence $\boldsymbol{p}$ is the point of M_i closest to the origin. Computing the length of $\boldsymbol{p}$ yields the formula given in the statement of the lemma. □

3. An upper bound

In this section we prove an upper bound

$$r(k,d) \leqslant \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{d-s}^2}{k-s}}, \tag{1}$$

where s is as in Theorem 1. The proof is by induction. The next lemma provides the base case for the induction.

Lemma 3. *Let $r(k,d)$ be the radius of the largest k-ball B inscribed inside a given d-dimensional box $\mathcal{B}$. Then*

$$r(k,d) \leqslant \sqrt{\frac{\sum_{i=1}^{d} a_i^2}{k}}.$$

Proof. By Lemma 1, we can assume B is centered at the origin. By Lemma 2, it follows that

$$r^2(k,d) = \min\left(a_1^2/T_1, a_2^2/T_2, \ldots, a_d^2/T_d\right).$$

Hence $r^2(k,d) \leqslant a_i^2/T_i$ and $r^2(k,d)T_i \leqslant a_i^2$. Thus $r^2(k,d)\sum_{i=1}^{d} T_i \leqslant \sum_{i=1}^{d} a_i^2$, which implies that $kr^2(k,d) \leqslant \sum_{i=1}^{d} a_i^2$ since

$$\sum_{i=1}^{d} T_i = \sum_{i=1}^{d}\sum_{j=1}^{k} v_{j,i}^2 = \sum_{j=1}^{k}\sum_{i=1}^{d} v_{j,i}^2 = \sum_{j=1}^{k} 1 = k. \qquad \square$$

Consider now the case when the k-ball is tangent to all facets of box $\mathcal{B}$. By replacing all occurances of $\leqslant$ by $=$ in the previous proof we obtain the following lemma.

Lemma 4. *If the largest k-ball B inside box $\mathcal{B}$ is tangent to all facets of $\mathcal{B}$ then*

$$r(k,d) = \sqrt{\frac{\sum_{i=1}^{d} a_i^2}{k}}.$$

The next lemma provides the inductive step for the proof of correctness of our upper bound for $r(k,d)$.

Notation. Let $r(k',d')$ be the radius of the largest k'-ball inside a d'-dimensional box $\mathcal{B}' \subseteq \mathcal{B}$ of size $2a_1 \times 2a_2 \times \cdots \times 2a_{d'}$, where $k' \leqslant k$ and $d' \leqslant d$. Note here that the dimensions of $\mathcal{B}'$ are equal to the corresponding dimensions of the original box $\mathcal{B}$.

Lemma 5. $r(k,d) \leqslant r(k-1,d-1)$.

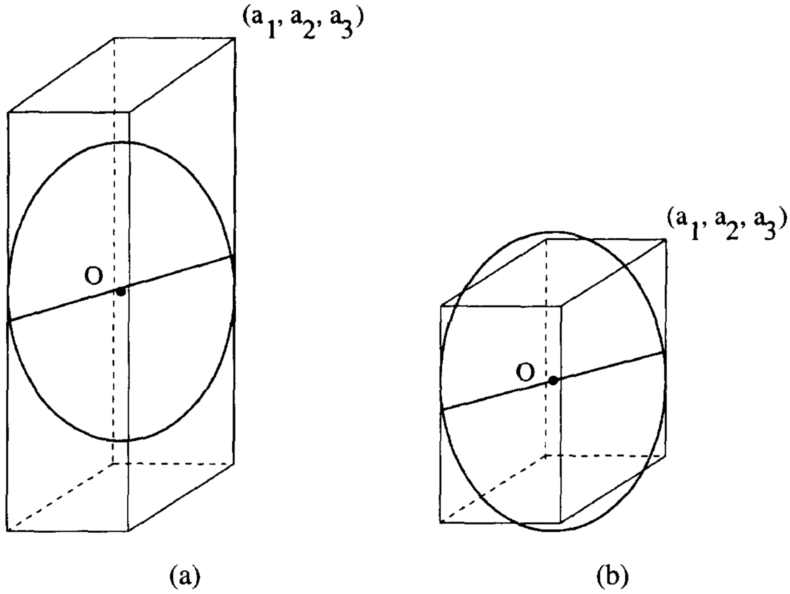

Fig. 2. (a) The box has room for the extension. (b) The extension does not fit.

Proof. Let B be a k-ball of maximum possible radius $r(k,d)$ centered at the origin and let M be the k-flat containing B. Consider flat $M' = M \cap \{x_d = 0\}$ of dimension $k-1$ or k. If dimension of M' is k, then $M' = M$ can be regarded as a k-dimensional flat in the $(d-1)$-dimensional box $\mathcal{B}'$, so $r(k,d) \leqslant r(k,d-1)$. Furthermore, $r(k,d-1) \leqslant r(k-1,d-1)$ since a k-ball in $(d-1)$-dimensional box $\mathcal{B}'$ gives rise to a $(k-1)$-ball of the same radius by intersecting the k-ball with a $(d-2)$-dimensional subspace not containing the k-ball. From now on, we assume that M' is a $(k-1)$-dimensional flat.

Every point in M' distance at most $r(k,d)$ from the origin belongs to $B \cap \{x_d = 0\}$. So, each point $\boldsymbol{x}' = (x_1, x_2, \ldots, x_{d-1})$ of $B \cap M'$ satisfies $|x_i| \leqslant |a_i|$ for $i = 1, 2, \ldots, d-1$. Therefore $B \cap \{x_d = 0\}$ is a $(k-1)$-dimensional ball of radius $r(k,d)$ which lies completely inside the box $\mathcal{B}'$. Thus $r(k,d) \leqslant r(k-1,d-1)$, as there may exist a larger ball inside $\mathcal{B}'$. □

Lemma 6. *If $a_d \geqslant r(k-1,d-1)$ then $r(k,d) = r(k-1,d-1)$.*

Proof. Let B' be a $(k-1)$-ball with radius $r(k-1,d-1)$ in $(d-1)$-dimensional box $\mathcal{B}'$. Hence each point $\boldsymbol{x}' = (x_1, x_2, \ldots, x_{d-1})$ from B' satisfies $x_1^2 + x_2^2 + \cdots + x_{d-1}^2 \leqslant r^2(k-1,d-1)$ and $|x_i| \leqslant |a_i|$ for $i = 1, 2, \ldots, d-1$. Let B be the set of points $(x_1, \ldots, x_d)$ with $(x_1, \ldots, x_{d-1}) \in B'$ and $x_1^2 + x_2^2 + \cdots + x_d^2 \leqslant r^2(k-1,d-1)$. See Fig. 2. Then B is a k-ball of radius $r(k-1,d-1)$. Since each point in B satisfies

$$x_d^2 \leqslant x_1^2 + x_2^2 + \cdots + x_d^2 \leqslant r^2(k-1,d-1) \leqslant a_d^2,$$

B is completely contained inside the box $\mathcal{B}$ of size $2a_1 \times 2a_2 \times \cdots \times 2a_d$. Hence $r(k,d) \geqslant r(k-1,d-1)$. The result now follows from the previous lemma. □

Notation. For future convenience, we define the following notation $b(k-s, d-s)$:

$$b(k-s, d-s) = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{d-s}^2}{k-s}},$$

where the parameter s satisfies $0 \leqslant s \leqslant k-1$.

Now, we are ready to complete the proof of (1).

Proof of (1). The proof is by backward induction on parameter s.

Note that the condition in the statement of Theorem 1 is $a_{d-s} \leqslant b(k-s-1, d-s-1)$. Also note that Lemma 3 implies $r(k-s, d-s) \leqslant b(k-s, d-s)$, for $s = 0, 1, \ldots, k-1$.

If $s = 0$ is the smallest integer that satisfies $a_{d-s} \leqslant b(k-s-1, d-s-1)$, then Lemma 3 gives (1).

Now suppose $s = 0$ does not satisfy $a_{d-s} \leqslant b(k-s-1, d-s-1)$. Then $a_d > b(k-1, d-1)$, which in turn is $\geqslant r(k-1, d-1)$ by Lemma 3. This implies by Lemma 6 that $r(k,d) = r(k-1, d-1)$, which is $\leqslant b(k-1, d-1)$ by Lemma 3. Thus if $a_{d-1} \leqslant b(k-2, d-2)$, then $s = 1$ is the smallest value for s and (1) holds.

Now suppose $a_{d-1} > b(k-2, d-2)$, which is $\geqslant r(k-2, d-2)$ by Lemma 3. But $a_{d-1} > r(k-2, d-2)$ implies by Lemma 6 that $r(k-1, d-1) = r(k-2, d-2)$. Hence

$$r(k,d) = r(k-1, d-1) = r(k-2, d-2),$$

which by Lemma 3 is $\leqslant b(k-2, d-2)$.

This process continues until the condition $a_{d-s} \leqslant b(k-s-1, d-s-1)$ is met, which must happen at some point since it is clearly satisfied for $s = k-1$. □

4. Tightness of the bound

In this section we complete the proof of Theorem 1 by showing that the upper bound of (1) is tight.

Suppose that, on the contrary, the bound is not tight. Let d be the minimum dimension for which the bound is not tight. Fix k and $\mathcal{B} \subset \mathbb{R}^d$ showing that the bound is not tight. If $a_d > b(k-1, d-1)$, then the value $s(k,d)$ for s in the formula (assumed inexact) for $r(k,d)$ must be greater than 0. It is easy to check that, as a consequence, the value for s in the formula (assumed exact) for $r(k-1, d-1)$ must be $s(k-1, d-1) = s(k,d) - 1$. Note that the value (exact) for $r(k-1, d-1)$ given by $s(k-1, d-1)$ is equal to the value for $r(k,d)$ (supposedly inexact) given by $s(k,d)$. A contradiction can now be reached as follows. Since $a_d > b(k-1, d-1)$, the largest $(k-1)$-dimensional ball B in a $(d-1)$-dimensional box can be "extended" as in the proof of Lemma 6 to a k-dimensional ball in d-space. The ball thus obtained has the same radius as B and lies in the k-flat containing B and the vector $(0, \ldots, 0, 1)$. This shows that the supposed inexact formula for $r(k,d)$ is in fact exact, a contradiction. Therefore, a_d must be equal to or less than $b(k-1, d-1)$.

Suppose that $a_d \leqslant b(k-1, d-1)$. This is exactly the condition of Theorem 1 for $s = 0$; thus $r(k,d) < b(k,d)$. From Lemma 4 it follows that the largest k-ball B is not tangent to all facets of $\mathcal{B}$. Suppose that B is not tangent to the mth facets, given by $x_m = +a_m$ and $x_m = -a_m$. We will now show that there exists a ball of radius greater than $r(k,d)$ inside the given box, which will contradict the maximality of $r(k,d)$. To do this, we will move B by a series of rotations so that B will no longer touch any of the facets of $\mathcal{B}$ and therefore will be enlargeable.

We produce the enlargeable k-ball from the original k-ball B by a series of $d-1$ rotations, some of which may be trivial. Beginning with $B_0 = B$, we produce a sequence of k-balls B_i, $0 \leqslant i \leqslant d$. For the special case $i = m$, we define $B_m = B_{m-1}$; otherwise, for $i \neq 0, i \neq m$, we adjust only the ith and the mth coordinates of points in k-ball B_{i-1}. We do this so that B_i has the same radius as B_{i-1} but does not intersect the ith facets of $\mathcal{B}$ and continues not to intersect the mth facets. Since only the mth and the ith coordinates are adjusted, tangency or non-tangency with respect to the remaining facets does not change.

The final k-ball B_d will be tangent to no facets and will have radius exactly equal to the formula for $r(k,d)$, contradicting the assumption that the formula gives a strict upper bound.

The transformations we are about to define can be understood simply in terms of their formulas, without reference to the notion of rotation in high dimension. However, we use the language of rotation for good reason (see [2]). In general, rotation in d dimensions is defined with respect to a $(d-2)$-dimensional flat. The action on a point p is defined by considering the 2-dimensional flat containing p that is orthogonal to the $(d-2)$-dimensional flat. Point p is rotated in this plane about the intersection point of the plane with the $(d-2)$-dimensional flat. In our case, we define rotations around $(d-2)$-flats of the form $\{x_m = 0, x_i = 0\}$.

We shall now define a rotation that takes B_{i-1} to B_i. If B_{i-1} is not tangent to the ith facets, we set $B_i = B_{i-1}$. Otherwise, consider the orthonormal basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ for the flat M containing the current k-ball, B_{i-1}. Each point from B_{i-1} is a linear combination $\boldsymbol{x} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_k\boldsymbol{v}_k$. The rotation of $\boldsymbol{x}$ can be obtained from rotations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$; that is, $R(\boldsymbol{x}) = R(\alpha_1\boldsymbol{v}_1 + \cdots + \alpha_k\boldsymbol{v}_k) = \alpha_1 R(\boldsymbol{v}_1) + \cdots + \alpha_k R(\boldsymbol{v}_k)$. The rotations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$, which affect only their ith and mth coordinates, are defined as follows:

$$v'_{j,m} = v_{j,m}\cos\theta - v_{j,i}\sin\theta, \qquad v'_{j,i} = v_{j,m}\sin\theta + v_{j,i}\cos\theta,$$

where θ is an angle of rotation and $R(\boldsymbol{v}_j) = \boldsymbol{v}'_j$, $1 \leqslant j \leqslant k$.

Define $M' = R(M)$, $M'_p = M' \cap \{x_p = a_p\}$ and $T'_p = \sum_{j=1}^{k} v'^2_{j,p}$, $1 \leqslant p \leqslant d$. From Lemma 2, the distance from the origin to the flat M'_p is $a_p/\sqrt{T'_p}$. But $v'_{j,p} = v_{j,p}$ for $p \neq m$, $p \neq i$ and any j. Thus no tangency properties other than for the ith and mth facets are affected by the rotation. Now

$$T'_m = \sum_{j=1}^{k} v'^2_{j,m} = \sum_{j=1}^{k} (v_{j,m}\cos\theta - v_{j,i}\sin\theta)^2 = T_m\cos^2\theta + T_i\sin^2\theta - 2\cos\theta\sin\theta\sum_{j=1}^{k} v_{j,m}v_{j,i}$$

and, similarly,

$$T'_i = \sum_{j=1}^{k} v'^2_{j,i} = \sum_{j=1}^{k} (v_{j,m}\sin\theta + v_{j,i}\cos\theta)^2 = T_m\sin^2\theta + T_i\cos^2\theta + 2\cos\theta\sin\theta\sum_{j=1}^{k} v_{j,m}v_{j,i}.$$

Thus, $T'_m + T'_i = T_m + T_i$.

Let $Q = \sum_{j=1}^{k} v_{j,m}v_{j,i}$. If the current k-ball B_{i-1} is tangent to the ith facets we want to increase the distance to M'_i, which can be done by decreasing T'_i (increasing T'_m at the same time). Since B_{i-1} is tangent to the ith facets and not to the mth facets, the origin is closer to M_i than to M_m. By Lemma 2, we have $T_m < T_i$. We now show how to choose θ so that $T'_i < T_i$, $T'_m < T_i$. We have

$$T_i - T'_i = T_i - \left(T_m\sin^2\theta + T_i\cos^2\theta + 2Q\cos\theta\sin\theta\right) = (T_i - T_m)\sin^2\theta - 2Q\sin\theta\cos\theta.$$

It follows that θ can be chosen (close to 0) so that

$$0 < T_i - T_i' < \Delta,$$

where $\Delta = T_i - T_m$. Then both T_i' and $T_m' = T_m + T_i - T_i'$ are smaller than T_i. It follows by Lemma 2 that B_i is tangent neither to the ith nor to the mth facets (nor to the jth facets, $j < i$).

Finally, B_d is tangent to no facet of $\mathcal{B}$, a contradiction with the maximality of B. This completes the proof of Theorem 1.

5. Conclusion and open problems

In this paper we have given a formula, in terms of the box dimensions, for the radius of the largest k-ball contained in a d-dimensional box.

Our result implies that the largest k-ball in a unit d-cube has diameter $\sqrt{d/k}$.

The value of s in Theorem 1 carries some information about the position of the largest k-ball(s) B in the box $\mathcal{B}$. If B is centered in the origin, then its position is fixed (up to the symmetries of $\mathcal{B}$). It is tangent to the first $d-s$ pairs of facets, and it can be translated within $\mathcal{B}$ by some distance in any direction parallel to the last s axes. Every largest k-ball in $\mathcal{B}$ is such a translate of one of the finitely many largest k-balls in $\mathcal{B}$ centered at the origin. However, we do not know how to determine exactly the position of the largest k-ball(s) B in the box $\mathcal{B}$. Our proof is not constructive, but perhaps it might be modified to get a fast algorithm for finding the largest k-ball(s) in $\mathcal{B}$.

We leave as an open problem to consider boxes that are parallelotopes whose adjacent sides are not necessarily perpendicular.

A related, widely open, problem is to determine the largest unit k-cube fitting into a unit d-cube. See [4, Section B4] and [7, pp. 967–968] for more information on this problem.

Another problem is to determine the largest k-volume of a k-ellipsoid fitting into a given d-box $\mathcal{B}$. As far as we know, this problem has not been considered yet.

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